On Learning, Teaching, and Learning Teaching

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ON LEARNING, TEACHING, AND LEARNING TEACHING

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"What you have been obliged to discover by yourself leaves a path in your mind which you can use again when the need arises." (G. C. Lichtenberg: *Aphorismen.*)

"Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas." (I. Kant: *Critique of Pure Reason*, translated by J. M. D. Meiklejohn, 1878, p. 429.)

"I [planned to] write so that the learner may always see the inner ground of the things he learns, even so that the source of the invention may appear, and therefore in such a way that the learner may understand everything as if he had invented it by himself." (G. W. von Leibnitz: *Mathematische Schriften*, edited by Gerhardt, vol. VII, p. 9.)

1. Teaching is not a science. I shall tell you some of my opinions on the process of learning, on the art of teaching, and on teacher training.

My opinions are the result of a long experience. Still, such personal opinions may be irrelevant and I would not dare to waste your time by telling them if teaching could be fully regulated by scientific facts and theories. This, however, is not the case. Teaching is, in my opinion, not just a branch of applied psychology—at any rate, it is not yet that for the present.

Teaching is correlated with learning. The experimental and theoretical study of learning is an extensively and intensively cultivated branch of psychology. Yet there is a difference. We are principally concerned here with complex learning situations, such as learning algebra or learning teaching, and their long-term educational effects. The psychologists, however, devote most of their attention to, and do their best work about, simplified short-term situations. Thus, the psychology of learning may give us interesting hints, but it can not pretend to pass ultimate judgment upon problems of teaching (cf. [1]).

2. The aim of teaching. We can not judge the teacher's performance if we do not know the teacher's aim. We can not meaningfully discuss teaching, if we do not agree to some extent about the aim of teaching.

Let me be specific. I am concerned here with mathematics in the high school curriculum and I have an old fashioned idea about its aim: first and foremost, it should teach those young people to THINK.

This is my firm conviction; you may not go along with it all the way, but I assume that you agree with it to some extent. If you do not regard "teaching to think" as a primary aim, you may regard it as a secondary aim—and then we have enough common ground for the following discussion.

"Teaching to think" means that the mathematics teacher should not merely impart information, but should try also to develop the ability of the students to use the information imparted: he should stress know-how, useful attitudes, desirable habits of mind. This aim may need fuller explanation (my whole printed work on teaching may be regarded as a fuller explanation) but here it will be enough to emphasize only two points.

First, the thinking with which we are concerned here is not day-dreaming but "thinking for a purpose" or "voluntary thinking" (William James) or "productive thinking" (Max Wertheimer). Such "thinking" may be identified here, at least in first approximation, with "problem solving." At any rate, in my opinion, one of the principal aims of the high school mathematics curriculum is to develop the students' ability to solve problems.

Second, mathematical thinking is not purely "formal"; it is not concerned only with axioms, definitions, and strict proofs, but many other things belong to it: generalizing from observed cases, inductive arguments, arguments from analogy, recognizing a mathematical concept in, or extracting it from, a concrete situation. The mathematics teacher has an excellent opportunity to acquaint his students with these highly important "informal" thought processes, and I mean that he should use this opportunity better, and much better, than he does today. Stated incompletely but concisely: Let us teach proving by all means, but let us also teach guessing.

3. Teaching is an art. Teaching is not a science, but an art. This opinion has been expressed by so many people so many times that I feel a little embarrassed repeating it. If, however, we leave a somewhat hackneyed generality and get down to appropriate particulars, we may see a few tricks of our trade in an instructive sidelight.

Teaching obviously has much in common with the theatrical art. For instance, you have to present to your class a proof which you know thoroughly having presented it already so many times in former years in the same course. You really can not be excited about this proof—but, please, do not show that to your class: if you appear bored, the whole class will be bored. Pretend to be excited about the proof when you start it, pretend to have bright ideas when you proceed, pretend to be surprised and elated when the proof ends. You should do a little acting for the sake of your students who may learn, occasionally, more from your attitudes than from the subject matter presented.

I must confess that I take pleasure in a little acting, especially now that I am old and very seldom find something new in mathematics: I may find a little satisfaction in re-enacting how I discovered this or that little point in the past.

Less obviously, teaching has something in common also with music. You know, of course, that the teacher should not say things just once or twice, but three or four or more times. Yet, repeating the same sentence several times without pause and change may be terribly boring and defeat its own purpose. Well, you can learn from the composers how to do it better. One of the principal art forms of music is "air with variations." Transposing this art form from music into teaching you begin by saying your sentence in its simplest form; then you repeat it with a little change; then you repeat it again with a little more color, and so on; you may wind up by returning to the original simple formulation.

Another musical art form is the "rondo." Transposing the rondo from music into teaching, you repeat the same essential sentence several times with little or no change, but you insert between two repetitions some appropriately contrasting illustrative material. I hope that when you listen the next time to a theme with variations by Beethoven or to a rondo by Mozart you will give a little thought to improving your teaching.

Now and then, teaching may approach poetry, and now and then it may approach profanity. May I tell you a little story about the great Einstein? I listened once to Einstein as he talked to a group of physicists in a party. "Why have all the electrons the same charge?" said he. "Well, why are all the little balls in the goat dung of the same size?" Why did Einstein say such things? Just to make some snobs to raise their eyebrows? He was not disinclined to do so, I think. Yet, probably, it went deeper. I do not think that the overheard remark of Einstein was quite casual. At any rate, I learnt something from it: Abstractions are important; use all means to make them more tangible. Nothing is too good or too bad, too poetical or too trivial to clarify your abstractions. As Montaigne put it: The truth is such a great thing that we should not disdain any means that could lead to it. Therefore, if the spirit moves you to be a little poetical, or a little profane, in your class, do not have the wrong kind of inhibition.

4. Three principles of learning. Teaching is a trade that has innumerable little tricks. Each good teacher has his pet devices and each good teacher is different from any other good teacher.

Any efficient teaching device must be correlated somehow with the nature of the learning process. We do not know too much about the learning process, but even a rough outline of some of its more obvious features may shed some welcome light upon the tricks of our trade. Let me state such a rough outline in the form of three "principles" of learning. Their formulation and combination is of my choice, but the "principles" themselves are by no means new; they have been stated and restated in various forms, they are derived from the experience of the ages, endorsed by the judgment of great men, and also suggested by the psychological study of learning.

These "principles of learning" can be also taken for "principles of teaching," and this is the chief reason for considering them here—but about this later.

(1) *Active learning.* It has been said by many people in many ways that learning should be active, not merely passive or receptive: merely by reading books or listening to lectures or looking at moving pictures without adding some action of your own mind you can hardly learn anything and certainly you can not learn much.

There is another often expressed (and closely related) opinion: *The best way to learn anything is to discover it by yourself.* Lichtenberg (an eighteenth century German physicist, better known as a writer of aphorisms) adds an interesting point: *What you have been obliged to discover by yourself leaves a path in your mind* *which you can use again when the need arises.* Less colorful is the following statement, but it may be more widely applicable: *For efficient learning, the learner should discover by himself as large a fraction of the material to be learnt as feasible under the given circumstances.*

This is the *principle of active learning* (Arbeitsprinzip). It is a very old principle: it underlies the idea of "Socratic method."

(2) *Best motivation.* Learning should be active, we have said. Yet the learner will not act if he has no motive to act. He must be induced to act by some stimulus, by the hope of some reward, for instance. The interest of the material to be learnt should be the best stimulus to learning and the pleasure of intensive mental activity should be the best reward for such activity. Yet, where we cannot obtain the best we should try to get the second best, or the third best, and less intrinsic motives of learning should not be forgotten.

For efficient learning, the learner should be interested in the material to be learnt and find pleasure in the activity of learning. Yet, beside these best motives for learning, there are other motives too, some of them desirable. (Punishment for not learning may be the least desirable motive.)

Let us call this statement the *principle of best motivation.*

(3) *Consecutive phases.* Let us start from an often quoted sentence of Kant: *Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas.* The English translation uses the terms "cognition, intuition, idea." I am not able (who is able?) to tell in what exact sense Kant intended to use these terms. Yet I beg your permission to present my reading of Kant's dictum:

Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits.

To begin with, please, take the terms of this sentence in some sense that you can illustrate concretely on the basis of your own experience. (To induce you to think about your personal experience is one of the desired effects.) "Learning" should remind you of a classroom with yourself in it as student or teacher. "Action and perception" should suggest manipulating and seeing concrete things such as pebbles, or apples, or Cuisenaire rods; or ruler and compasses; or instruments in a laboratory; and so on.

Such concrete interpretation of the terms may come more easily and more naturally when we think of some simple elementary material. Yet after a while we may perceive similar phases in the work spent on mastering more complex, more advanced material. Let us distinguish three phases: the phases of *exploration, formalization,* and *assimilation.*

A first *exploratory* phase is closer to action and perception and moves on a more intuitive, more heuristic level.

A second *formalizing* phase ascends to a more conceptual level, introducing terminology, definitions, proofs.

The phase of *assimilation* comes last: there should be an attempt to perceive the "inner ground" of things, the material learnt should be mentally digested, absorbed into the system of knowledge, into the whole mental outlook of the learner; this phase paves the way to applications on one hand, to higher generalizations on the other.

Let us summarize: *For efficient learning, an exploratory phase should precede the phase of verbalization and concept formation and, eventually, the material learnt should be merged in, and contribute to, the integral mental attitude of the learner.*

This is the *principle of consecutive phases.*

5. Three principles of teaching. The teacher should know about the ways of learning. He should avoid inefficient ways and take advantage of the efficient ways of learning. Thus, he can make good use of the three principles we have just surveyed, the principle of active learning, the principle of best motivation, and the principle of consecutive phases: these principles of learning are also principles of teaching. There is, however, a condition: to avail himself of such a principle, the teacher should not merely know it from hearsay, but he should understand it intimately on the basis of his own well-considered personal experience.

(1) *Active learning.* What the teacher says in the classroom is not unimportant, but what the students think is a thousand times more important. The ideas should be born in the students' mind and the teacher should act only as midwife.

This is a classical Socratic precept and the form of teaching best adapted to it is the Socratic dialogue. It is a definite advantage of the high school teacher over the college instructor that in the high school one can use the dialogue form much more extensively than in the college. Unfortunately, even in the high school, time is limited and there is a prescribed material to cover so that all business cannot be transacted in dialogue form. Yet the principle is: Let the students *discover by themselves as much as feasible* under the given circumstances.

Much more is feasible than is usually done, I am sure. Let me recommend you here just one little practical trick: Let the students *actively contribute to the formulation* of the problem that they have to solve afterwards. If the students have had a share in proposing the problem they will work at it much more actively afterwards.

In fact, in the work of the scientist, formulating the problem may be the better part of a discovery, the solution often needs less insight and originality than the formulation. Thus, letting your students have a share in the formulation, you not only motivate them to work harder, but you teach them a desirable attitude of mind.

(2) *Best motivation.* The teacher should regard himself as a salesman: he wants to sell some mathematics to the youngsters. Now, if the salesman meets with sales resistance and his prospective customers refuse to buy, he should not lay the whole blame on them. Remember, the customer is always right in principle, and sometimes right in practice. The lad who refuses to learn mathematics may be right: he may be neither lazy nor stupid, just more interested in

something else—there are so many interesting things in the world around us. It is your duty as a teacher, as a salesman of knowledge, to convince the student that mathematics *is* interesting, that the point just under discussion is interesting, that the problem he is supposed to do deserves his effort.

Therefore, the teacher should pay attention to the choice, the formulation, and a suitable presentation of the problem he proposes. The problem should be related, if possible, to the everyday experience of the students, and it should be introduced, if possible, by a little joke or a little paradox. Or the problem should start from some very familiar knowledge; it should have, if possible, some point of general interest or eventual practical use. If we wish to stimulate the student to a genuine effort, we must give him some reason to suspect that his task deserves his effort.

The best motivation is the student's interest in his task. Yet there are other motivations which should not be neglected. Let me recommend here just one little practical trick. Before the students do a problem, let them *guess the result*, or a part of the result. The boy who expresses an opinion commits himself; his prestige and self-esteem depend a little on the outcome, he is impatient to know whether his guess will turn out right or not, and so he will be actively interested in his task and in the work of the class—he will not fall asleep or misbehave.

In fact, in the work of the scientist, the guess almost always precedes the proof. Thus, in letting your students guess the result, you not only motivate them to work harder, but you teach them a desirable attitude of mind.

(3) *Consecutive phases.* The trouble with the usual problem material of the high school textbooks is that they contain almost exclusively merely routine examples. A routine example is a short range example; it illustrates, and offers practice in the application of, just one isolated rule. Such routine examples may be useful and even necessary, I do not deny it, but they miss two important phases of learning: the exploratory phase and the phase of assimilation. Both phases seek to connect the problem in hand with the world around us and with other knowledge, the first before, the last after, the formal solution. Yet the routine problem is obviously connected with the rule it illustrates and it is scarcely connected with anything else, so that there is little profit in seeking further connections. In contrast with such routine problems, the high school should present more challenging problems at least now and then, problems with a rich background that deserves further exploration, and problems which can give a foretaste of the scientist's work.

Here is a practical hint: if the problem you want to discuss with your class is suitable, let your students do some preliminary exploration: it may whet their appetite for the formal solution. And reserve some time for a retrospective discussion of the finished solution; it may help in the solution of later problems.

(4) After this much too incomplete discussion, I must stop explaining the three principles of active learning, best motivation, and consecutive phases. I think that these principles can penetrate the details of the teacher's daily work and make him a better teacher. I think too that these principles should also

penetrate the planning of the whole curriculum, the planning of each course of the curriculum, and the planning of each chapter of each course.

Yet it is far from me to say that you must accept these principles. These principles proceed from a certain general outlook, from a certain philosophy, and you may have a different philosophy. Now, in teaching as in several other things, it does not matter much what your philosophy is or is not. It matters more whether you have a philosophy or not. And it matters very much whether you try to live up to your philosophy or not. The only principles of teaching which I thoroughly dislike are those to which people pay only lip service.

6. Examples. Examples are better than precepts; let me get down to examples—I much prefer examples to general talk. I am here concerned principally with teaching on the high school level and I shall present you a few examples on that level. I often find satisfaction in treating examples at the high school level, and I can tell you why: I attempt to treat them so that they recall in one respect or the other my own mathematical experience; I am re-enacting my past work on a reduced scale.

(1) *A seventh grade problem.* The fundamental art form of teaching is the Socratic dialogue. In a junior high school class, perhaps in the seventh grade, the teacher may start the dialogue so:

"What is the time at noon in San Francisco?"

'But, teacher everybody knows that' may say a lively youngster, or even 'But teacher, you are silly: twelve o'clock.'

"And what is the time at noon in Sacramento?"

'Twelve o'clock—of course, not twelve o'clock midnight.'

"And what is the time at noon in New York?"

'Twelve o'clock.'

"But I thought that San Francisco and New York do not have noon at the same time, and you say that both have noon at twelve o'clock!"

'Well, San Francisco has noon at twelve o'clock Western Standard Time and New York at twelve o'clock Eastern Standard Time.'

"And on what kind of standard time is Sacramento, Eastern or Western?"

'Western, of course.'

"Have the people in San Francisco and Sacramento noon at the same moment?"

"You do not know the answer? Well, try to guess it: does noon come sooner to San Francisco, or to Sacramento, or does it arrive exactly at the same instant at both places?"

How do you like my idea of Socrates talking to seventh grade kids? At any rate, you can imagine the rest. By appropriate questions the teacher, imitating Socrates, should extract several points from the students:

(a) We have to distinguish between "astronomical" noon and conventional or "legal" noon.

(b) Definitions for the two noons.

(c) Understanding "standard time": how and why is the globe's surface subdivided into time zones?

(d) Formulation of the problem: "At what o'clock Western Standard Time is the astronomical noon in San Francisco?"

(e) The only specific datum needed to solve the problem is the longitude of San Francisco (in an approximation sufficient for the seventh grade).

The problem is not too easy. I tried it on two classes; in both classes the participants were high school teachers. One class spent about 25 minutes on the solution, the other 35 minutes.

(2) I must say that this little seventh grade problem has various advantages. Its main advantage may be that it emphasizes an essential mental operation which is sadly neglected by the usual problem material of the textbooks: *recognizing the essential mathematical concept in a concrete situation.* To solve the problem, the students must recognize a *proportionality*: the time of the highest position of the sun in a locality on the globe's surface changes *proportionally* to the longitude of the locality.

In fact, in comparison with the many painfully artificial problems of the high school textbooks, our problem is a perfectly natural, a "real" problem. In the serious problems of applied mathematics, the appropriate *formulation* of the problem is always a major task, and often the most important task; our little problem which can be proposed to an average seventh grade class possesses just this feature. Again, the serious problems of applied mathematics may lead to practical action, for instance, to adopting a better manufacturing process; our little problem can explain to seventh graders why the system of 24 time zones, each with a uniform standard time, was adopted. On the whole, I think that this problem, if handled with a little skill by the teacher, could help a future scientist or engineer to discover his vocation, and it could also contribute to the intellectual maturity of those students who will not use mathematics professionally.

Observe also that this problem illustrates several little tricks mentioned in the foregoing: The students actively contribute to the formulation of the problem (cf. Sect. 5(1)). In fact, the exploratory phase which leads to the formulation of the problem is prominently important (cf. Sect. 5(3)). Then, the students are invited to guess an essential point of the solution (cf. Sect. 5(2)).

(3) *A tenth grade problem.* Let us consider another example. Let us start from what is probably the most familiar problem of geometric construction: *Construct a triangle, being given its three sides.* As analogy is such a fertile source of invention, it is natural to ask: What is the analogous problem in solid geometry? An average student, who has a little knowledge of solid geometry, may be led to formulate the problem: *Construct a tetrahedron, being given its six edges.*

It may be mentioned here parenthetically that this problem of the tetrahedron comes as close as it can on the usual high school level to practical problems solved by "mechanical drawing." Engineers and designers use well executed drawings to give precise information about the details of three dimensional figures of machines or structures to be built: we intend to build a tetrahedron with specified edges. We might wish, for example, to carve it out of wood.

This leads to asking that the problem should be solved precisely, by straightedge and compasses, and to discussing the question: which details of the tetrahedron should be constructed? Eventually, from a well conducted class discussion, the following definitive formulation of the problem may emerge:

Of the tetrahedron ABCD, we are given the lengths of its six edges

$$AB,\ BC,\ CA,\ AD,\ BD,\ CD.$$

Regard $\triangle ABC$ as the base of the tetrahedron and construct with ruler and compasses the angles that the base includes with the other three faces.

The knowledge of these angles is required for cutting out of wood the desired solid. Yet other elements of the tetrahedron may turn up in the discussion such as

(a) the altitude drawn from the vertex D opposite the base,
(b) the foot F of this altitude in the plane of the base;

(a) and (b) would contribute to the knowledge of the solid, they may possibly help to find the required angles, and so we may try to construct them too.

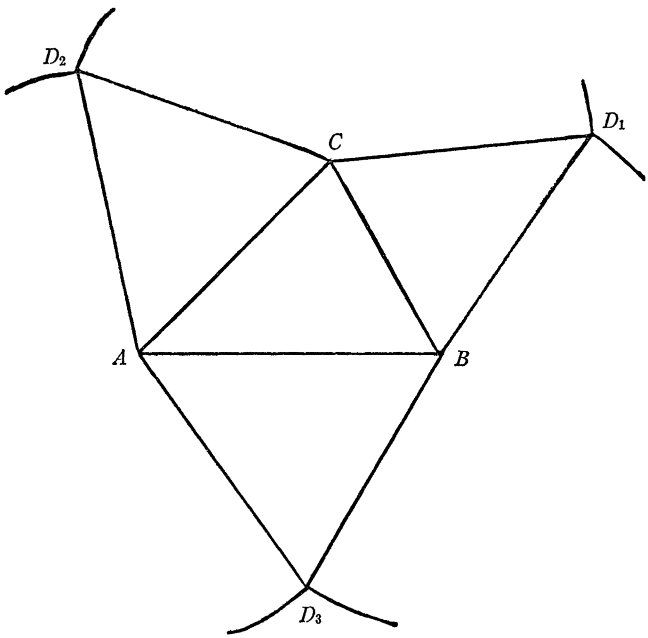

FIG. 1. Tetrahedron from six edges.

(4) We can, of course, construct the four triangular faces which are assembled in Fig. 1. (Short portions of some circles used in the construction are preserved to indicate that $AD_2=AD_3$, $BD_3=BD_1$, $CD_1=CD_2$.) If Fig. 1 is copied

on cardboard, we can add three flaps, cut out the pattern, fold it along three lines, and paste down the flaps; we obtain in this way a solid model on which we can measure roughly the altitude and the angles in question. Such work with cardboard is quite suggestive, but it is not what we are required to do: we should construct the altitude, its foot, and the angles in question with ruler and compasses.

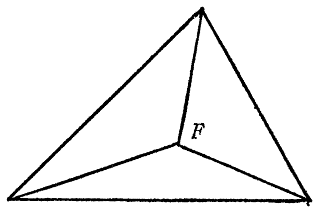

FIG. 2. An aspect of the finished product.

(5) It may help to take the problem, or some part of it, "as solved." Let us visualize how Fig. 1 will look when the three lateral faces, after having been rotated each about a side of the base, will be lifted into their proper position. Fig. 2 shows the orthogonal projection of the tetrahedron onto the plane of its base, $\triangle ABC$. The point F is the projection of the vertex D: it is the foot of the altitude drawn from D.

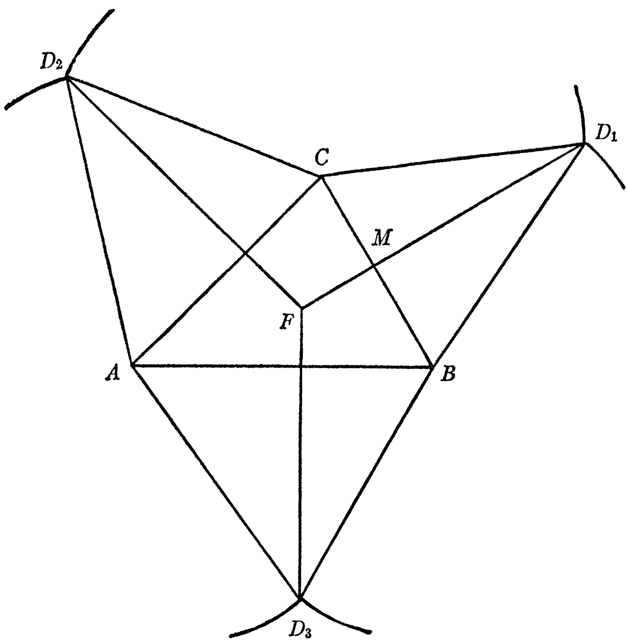

FIG. 3. The common destination of three travelers.

(6) We may visualize the transition from Fig. 1 to Fig. 2 with or without a cardboard model. Let us focus our attention upon one of the three lateral faces, upon $\triangle BCD_1$, which was originally located in the same plane as $\triangle ABC$, in the plane of Fig. 1 which we imagine as horizontal. Let us watch the triangle BCD_1 rotating about its fixed side BC and let our eyes follow its only moving vertex D_1. This vertex D_1 describes an arc of a circle. The center of this circle is a point of BC; the plane of this circle is perpendicular to the horizontal axis of revolution BC; thus D_1 moves in a vertical plane. Therefore, the projection of the path of the moving vertex D_1 onto the horizontal plane of Fig. 1 is a straight line, perpendicular to BC, passing through the original position of D_1.

Yet there are two more rotating triangles, three altogether. There are three moving vertices, each following a circular path in a vertical plane—to which destination?

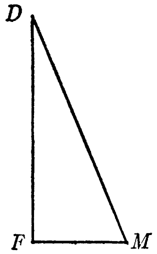

FIG. 4. The rest is easy.

(7) I think that by now the reader has guessed the result (perhaps even before reading the end of the foregoing subsection): the three straight lines drawn from the original positions (see Fig. 1) of D_1, D_2, and D_3 perpendicularly to BC, CA, and AB, respectively, meet in one point, the point F, our supplementary aim (b), see Fig. 3. (It is enough to draw two perpendiculars to determine F, but we may use the third to check the precision of our drawing.) And what remains to do is easy. Let M be the point of intersection of D_1F and BC (see Fig. 3). Construct the right triangle FMD (see Fig. 4), with hypotenuse $MD=MD_1$ and leg MF. Obviously, FD is the altitude (our supplementary aim (a)) and $\angle FMD$ measures the dihedral angle included by the base $\triangle ABC$ and the lateral face $\triangle DBC$ which was required by our problem.

(8) One of the virtues of a good problem is that it generates other good problems.

The foregoing solution may, and should, leave a doubt in our mind. We found the result represented by Fig. 3 (that the three perpendiculars described above are concurrent) by considering the motion of rotating bodies. Yet the result is a proposition of geometry and so it should be established independently of the idea of motion, by geometry alone.

Now, it is relatively easy to free the foregoing consideration (in subsections (6) and (7)) from ideas of motion and establish the result by ideas of solid geometry (intersection of spheres, orthogonal projection). Yet the result is a proposition of plane geometry and so it should be established independently of the ideas of solid geometry, by plane geometry alone. (How?)

(9) Observe that this tenth grade problem also illustrates various points about teaching discussed in the foregoing. For instance, the students could and should participate in the final formulation of the problem, there is an exploratory phase, and a rich background.

Yet here is the point I wish to emphasize: the problem is designed to deserve the attention of the students. Although the problem is not so close to everyday experience as our seventh grade problem, it starts from a most familiar piece of knowledge (the construction of a triangle from three sides) it stresses from the start an idea of general interest (analogy) and it points to eventual practical applications (mechanical drawing). With a little skill and good will, the teacher should be able to secure for this problem the attention of all students who are not hopelessly dull.

7. Learning teaching. There remains one more topic to discuss and it is an important topic: teacher training. In discussing this topic, I am in a comfortable position: I can almost agree with the "official" standpoint. (I am referring here to the "Recommendations of the Mathematical Association of America for the training of mathematics teachers," this MONTHLY, 67 (1960) 982–991. Just for the sake of brevity, I take the liberty to quote this document as the "official recommendations.") I shall concentrate upon just two points. To these two points I have devoted a good deal of work and thought in the past and practically all my teaching in the last ten years.

To state it roughly, one of the two points I have in mind is concerned with "subject matter" courses, the other with "methods" courses.

(1) *Subject matter.* It is a sad fact, but by now widely recognized, that our high school mathematics teachers' knowledge of their subject matter is, on the average, insufficient. There are, certainly, some well-prepared high school teachers, but there are others (I met with several) whose good will I must admire but whose mathematical preparation is not admirable. The official recommendations of subject matter courses may not be perfect, but there is no doubt that their acceptance would result in substantial improvement. I wish to direct your attention to a point which, in my considered opinion, should be added to the official recommendations.

Our knowledge about any subject consists of information and know-how. Know-how is ability to use information; of course, there is no know-how without some independent thinking, originality, and creativity. Know-how in mathematics is the ability to do problems, to find proofs, to criticize arguments, to use mathematical language with some fluency, to recognize mathematical concepts in concrete situations.

Everybody agrees that, in mathematics, know-how is more important, or even much more important, than mere possession of information. Everybody demands that the high school should impart to the students not only information in mathematics but know-how, independence, originality, creativity. Yet almost nobody asks these beautiful things for the mathematics teacher—is it not remarkable? The official recommendations are silent about the mathematical know-how of the teacher. The student of mathematics who works for a Ph.D. degree must do research, yet even before he reaches that stage he may find some opportunity for independent work in seminars, problem seminars, or in the preparation of a master's thesis. Yet no such opportunity is offered to the prospective mathematics teacher—there is no word about any sort of independent work or research work in the official recommendations. If, however, the teacher has had no experience of creative work of some sort, how will he be able to inspire, to lead, to help, or even to recognize the creative activity of his students? A teacher who acquired whatever he knows in mathematics purely receptively can hardly promote the active learning of his students. A teacher who never had a bright idea in his life will probably reprimand a student who has one instead of encouraging him.

Here, in my opinion, is the worst gap in the subject matter knowledge of the average high school teacher: he has no experience of active mathematical work and, therefore, he has no real mastery even of the high school material he is supposed to teach.

I have no panacea to offer, but I have tried one thing. I have introduced and repeatedly conducted a *problem solving seminar* for teachers. The problems offered in this seminar do not require much knowledge beyond the high school level, but they require some degree, and now and then a higher degree, of concentration and judgment—and, to that degree, their solution is "creative" work. I have tried to arrange my seminar so that the students should be able to use much of the material offered in their classes without much change; that they should acquire some mastery of high school mathematics; and so that they should have even some opportunity for practice teaching (in teaching each other in small groups). I can not enter here upon details; I gave a detailed description in a recently published book [2].

(2) *Methods.* From my contact with hundreds of mathematics teachers I gained the impression that "methods" courses are often received with something less than enthusiasm. Yet so also are received, by the teachers, the usual courses offered by the mathematics departments. A teacher with whom I had a heart to heart talk about these matters found a picturesque expression for a rather widespread feeling: "The mathematics department offers us tough steak which we can not chew and the school of education vapid soup with no meat in it."

In fact, we should once summon up some courage and discuss publicly the question: Are methods courses really necessary? Are they in any way useful? There is more chance to reach the right answer in open discussion than by widespread grumbling.

There are certainly enough pertinent questions. Is teaching teachable? (Teaching is an art, as many of us think—is an art teachable?) Is there such a thing as the teaching method? (What the teacher teaches is never better than what the teacher is—teaching depends on the whole personality of the teacher—there are as many good methods as there are good teachers.) The time allotted to the training of teachers is divided between subject matter courses, methods courses, and practice teaching; should we spend less time on methods courses? (Many European countries spend much less time.)

I hope that people younger and more vigorous than myself will take up these questions some day and discuss them with an open mind and pertinent data.

I am speaking here only about my own experience and my own opinions. In fact, in this hour, I have already implicitly answered the main question raised: I believe that methods courses may be useful. In fact, what I have presented to you in this hour was a sample of a methods course, or rather an outline of some topics which, in my opinion, a methods course offered to mathematics teachers should cover.

In fact, all the classes I have given to mathematics teachers were intended to be methods courses to some extent. The name of the class mentioned some subject matter, and the time was actually divided between that subject matter and methods: perhaps nine tenths for subject matter and one tenth for methods. If possible, the class was conducted in dialogue form. Some methodical remarks were injected incidentally, by myself or by the audience. Yet the derivation of a fact or the solution of a problem was almost regularly followed by a short discussion of its pedagogical implications. "Could you use this in your classes?" I asked the audience. "At which stage of the curriculum could you use it? Which point needs particular care? How would you try to get it across?" And questions of this nature (appropriately specified) were regularly proposed also in examination papers. My main work was, however, to choose such problems (like the two problems I have here presented) as would illustrate strikingly some pattern of teaching.

(3) The official recommendations call "methods" courses "curriculum-study" courses and are not very eloquent about them. Yet you can find there one recommendation that is excellent, I think. It is somewhat concealed; you must put two and two together, combining the last sentence in "curriculum study courses" and the recommendations for Level IV. But it is clear enough: A college instructor who offers a methods course to mathematics teachers should know mathematics at least on the level of a Master's degree. I would like to add: he should also have had some experience, however modest, of mathematical research. If he had no such experience how could he convey what may be the most important thing for prospective teachers, the spirit of creative work?

You have now listened long enough to the reminiscences of an old man. Some concrete good could come out of this talk if you give some thought to the following proposal which results from the foregoing discussion. I propose that the following two points should be added to the official recommendations of the Association:

I. *The training of teachers of mathematics should offer experience in independent ("creative") work on the appropriate level in the form of a Problem Solving Seminar or in any other suitable form.*

II. *Methods courses should be offered only in close connection either with subject matter courses or with practice teaching and, if feasible, only by instructors experienced both in mathematical research and in teaching.*

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ORDERED GROUPS

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1. Introduction. One of the more recent movements in pure mathematics (see [1], pp. 214–237) has been concerned with the "remarriage" of algebraic systems based upon the properties of certain operations to those based upon order relations. The union is achieved by means of an axiom or axioms relating the operation(s) to the order relation. It is interesting to note that, appearing as special cases, we find such systems as the integers and real numbers whose order relations and algebraic operations were at one time "divorced" in order that venturesome mathematicians might freely explore the delights of each.

An important example is the *partially ordered group*, a system P satisfying (i) P is a group under the operation $+$; (ii) P is a partially ordered set under the relation $\geqq$; and (iii) if a and b are elements in P such that $a \geqq b$, then $x+a+y \geqq x+b+y$ for any pair of elements x and y in P. Thinking of ways in which this study might be extended, Frink proposed a definition of an *ideal* in a partially ordered set and suggested that one might consider systems in which the algebraic operations preserve ideals (see [2]). We are to explore the consequences of this and other generalizations of postulate (iii) while leaving the very general axioms, (i) and (ii), practically unchanged.

2. Preliminary definitions.

DEFINITION 1. *If F is a nonvoid subset of a partially ordered set P, F^* denotes the set of all elements x of P such that $x \geqq f$ for every element f of F, and F^+ denotes*

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